INVARIANT SUBSPACES WITH FINITE CODIMENSION IN BERGMAN SPACES

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ABSTRACT. For an arbitrary bounded domain in $\mathbb{C}$ there are described those finite codimensional subspaces of the Bergman space that are invariant under multiplication by z.

1. INTRODUCTION

Let Ω be a domain in the complex plane. An analytic function f in Ω belongs to the Bergman space $L_a^p(\Omega)$, $1 \le p < \infty$, if

$$(1.1) \qquad \|f\|_p = \left(\int_\Omega |f|^p \, dm \right)^{1/p} < \infty,$$

where m is the area measure on $\mathbb{C}$. It is well known that for all $p \ge 1$, $L_a^p(\Omega)$ is a closed subspace of $L^p(\Omega, m)$. Let z be the identity function on Ω and M_z the linear operator on $L_a^p(\Omega)$ given by multiplication by z, i.e.

$$(1.2) \qquad M_z f = zf, \qquad f \in L_a^p(\Omega).$$

If Ω is bounded, M_z is a bounded linear operator on $L_a^p(\Omega)$. The aim of this paper is to describe all closed invariant subspaces of the operator M_z which have finite codimension in $L_a^p(\Omega)$. This problem was recently solved by S. Axler and P. Bourdon [3] in the case when every connected component of $\partial\Omega$, the boundary of Ω, contains more than one point. They showed that for such bounded domains Ω these subspaces have the form $QL_a^p(\Omega)$, where Q is a polynomial whose zeros lie in Ω. The structure of finite codimensional invariant subspaces of $L_a^p(\Omega)$ is more complicated for arbitrary bounded domains Ω, especially in the case $p \ge 2$. However it will turn out from the results proved in §3 of this paper that each closed invariant subspace E of finite codimension in $L_a^p(\Omega)$ has the form $E = QL_a^p(\Omega)$, if $1 \le p < 2$, or $E = [QL_a^p(\Omega)]^-$ if $p \ge 2$, where Q is a polynomial whose degree equals the codimension of E and whose zeros lie in Ω^-. Moreover, if $1 \le p < 2$, the zeros of Q are either points of Ω or isolated points of $\partial\Omega$. For $p \ge 2$, the location of the zeros of Q is a more delicate problem. Some examples and further results in this direction are deferred to §4. The characterization of finite codimensional invariant subspaces of M_z provides positive answers to several questions raised

in [3]. The main tool for the proof of the results mentioned above is a local approximation theorem for Bergman spaces, which is contained in §2.

2. AN APPROXIMATION THEOREM

We begin with the definition of the so-called localization operators on $L_a^p(\Omega)$, which will play an essential role in what follows.

Let φ be a real-valued function in $C^1[0, \infty)$ with the following properties:

$$(2.1) \qquad \varphi(x) = 1 \quad \text{if } 0 \leq x \leq 1 \quad \text{and} \quad \varphi(x) = 0 \quad \text{if } x \geq 2.$$

$$(2.2) \qquad |\varphi'(x)| \leq 2, \qquad x \in [0, \infty).$$

Let Ω be a bounded domain in $\mathbb{C}$ and $\lambda \in \partial\Omega$. Further, for $p \geq 1$ extend each function in $L_a^p(\Omega)$ to $\mathbb{C}$ by letting $f(u) = 0$ if $u \notin \Omega$. For $\delta > 0$ let $\varphi_\delta(u) = \varphi(|u - \lambda|/\delta)$ and consider the linear operator T_δ on $L_a^p(\Omega)$ given by

$$(2.3) \qquad (T_\delta f)(u) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(u) - f(v)}{u - v} \frac{\partial \varphi_\delta}{\partial \overline{z}}(v) \, dm(v), \qquad f \in L_a^p(\Omega).$$

The function $T_\delta f$ is actually defined a.e. in $\mathbb{C}$ and has the following properties:

$$(2.4) \qquad T_\delta f \text{ is analytic in } \Omega \cup \{u \colon |u - \lambda| > 2\delta\} \quad \text{and} \quad T_\delta f(\infty) = 0,$$

$$(2.5) \qquad f - T_\delta f \text{ is analytic in } \{u \colon |u - \lambda| < \delta\}.$$

These properties are well known in the case when f is bounded and follow by an application of Green's formula [4, p. 29]. It is not hard to see that (2.4) and (2.5) hold for $f \in L_a^p(\Omega)$ as well. Moreover, $T_\delta f$ is the convolution of the locally integrable function z^{-1} and a compactly supported L^p-function, which shows that T_δ is a bounded linear operator on $L_a^p(\Omega)$.

For $p \geq 1$ and $\lambda \in \partial\Omega$, denote by $A_\Omega^p(\lambda)$ the set of functions $f \in L_a^p(\Omega)$ with the property that there exists a neighborhood $V = V(f)$ of λ such that f extends analytically to $\Omega \cup V$. The main result of this section is

2.1. Theorem. *Let $\lambda \in \partial\Omega$. Then*

    (i) *For $p \geq 2$, $A_\Omega^p(\lambda)$ is dense in $L_a^p(\Omega)$,*
    (ii) *For $1 \leq p < 2$, $(z - \lambda)^{-1} A_\Omega^p(\lambda)$ is dense in $L_a^p(\Omega)$.*

For the proof we need the following lemma.

2.2. Lemma. *Let $1 < p \leq \infty$ and f, g be nonnegative functions on $\mathbb{C}$ with $f \in L^1(\mathbb{C}, m)$ and $g \in L^p(\mathbb{C}, m)$. If $1/p + 1/q = 1$ then*

$$(2.6) \qquad \liminf_{\delta \to 0} \left(\int_{\delta < |u| < 2\delta} f(u) \, dm \right)^{1/q} \cdot \int_{|u| > \delta} g(u) \cdot |u|^{-2/q} \, dm = 0.$$

Proof. Denote by c the limit in (2.6) and assume that $c > 0$. If $0 < c_1 < c$, there exists $\delta \in (0, 1)$ such that for all $\varepsilon \leq \delta$ we have

$$(2.7) \qquad c_1^q \leq \int_{\varepsilon < |u| < 2\varepsilon} f(u) \, dm \cdot \left(\int_{\varepsilon < |u| < 1} g(u) |u|^{-2/q} \, dm \right)^q$$

$$\leq \|g\|_p^q \log \frac{1}{\varepsilon} \int_{\varepsilon < |u| < 2\varepsilon} f(u) \, dm,$$

by Hölder's inequality. Letting $\varepsilon_k = 2^{-k}\delta$, $k \geq 1$, from (2.7) we obtain

$$(2.8) \qquad c_1^q \cdot \sum_{k=1}^{n} \left(\log\frac{1}{\delta} + k\log 2 \right)^{-1}$$

$$\leq \|g\|_p^q \cdot \sum_{k=1}^{n} \int_{2^{-k}\delta < |u| < 2^{-k+1}\delta} f(u)\, dm \leq \|g\|_p^q \cdot \|f\|_1 \,,$$

for all $n \geq 1$. This is a contradiction which shows that $c = 0$.

Proof of Theorem 2.1. The proof will follow by several applications of Hölder's inequality. There is no loss of generality if we assume that $\lambda = 0$. Let $x \in [A_\Omega^p(0)]^\perp$ if $p \geq 2$, or $x \in [z^{-1}A_\Omega^p(0)]^\perp$ if $1 \leq p < 2$. By the Hahn-Banach theorem, there exists $h \in L^q(\Omega, m)$, $1/p + 1/q = 1$, such that $x(f) = \int_\Omega hf\,dm$, $f \in L_a^p(\Omega)$. For such f, by (2.4) and (2.5) we have

$$(2.9) \qquad \int_\Omega hf\,dm = \int_\Omega h \cdot T_\delta f\,dm, \qquad \delta > 0.$$

Using (2.3), Green's formula and Fubini's theorem we obtain

$$(2.10) \quad \int_\Omega hf\,dm = \int_\Omega hf\varphi_\delta\,dm - \frac{1}{\pi}\int_\Omega f(v)\frac{\partial\varphi_\delta}{\partial\overline{z}}(v)\left(\int_\Omega \frac{h(u)}{u-v}\,dm(u) \right) dm(v).$$

Since φ_δ is supported on the set $\{u\colon \delta \leq |u| \leq 2\delta\}$ and $|\partial\varphi_\delta/\partial\overline{z}| \leq 2\delta^{-1}$, the usual estimation of the L^p-norm of a convolution gives

$$(2.11) \qquad \lim_{\delta\to 0}\frac{1}{\pi}\int_\Omega f(v)\frac{\partial\varphi_\delta}{\partial\overline{z}}(v)\left(\int_{|u|<3\delta} \frac{h(u)}{u-v}\,dm(u) \right) dm(v) = 0,$$

which leads to
$$(2.12)$$
$$\int_\Omega hf\,dm = -\frac{1}{\pi}\lim_{\delta\to 0}\int_{\delta<|v|<2\delta} f(v)\frac{\partial\varphi_\delta}{\partial\overline{z}}(v)\left(\int_{|u|>3\delta} \frac{h(u)}{u-v}\,dm(u) \right) dm(v).$$

By a series development we obtain

$$(2.13) \qquad \int_\Omega hf\,dm = -\frac{1}{\pi}\lim_{\delta\to 0}\sum_{k=0}^{\infty} a_k(\delta)b_k(\delta),$$

where $a_k(\delta)$, $b_k(\delta)$ are given by

$$(2.14) \qquad a_k(\delta) = \int_{\delta<|v|<2\delta} f(v)v^k\frac{\partial\varphi_\delta}{\partial\overline{z}}(v)\,dm,$$

and

$$(2.15) \qquad b_k(\delta) = \int_{|u|>3\delta} h(u)\cdot u^{-k-1}\,dm,$$

for all $k \geq 0$. Assume first that $p > 2$. Using again the estimation $|\partial\varphi_\delta/\partial\overline{z}| \leq 2\delta^{-1}$ and Hölder's inequality we obtain for all $\delta > 0$

$$(2.16) \qquad |a_k(\delta)| \leq c_p(2\delta)^{k+1-2/p} \cdot \left(\int_{\delta<|v|<2\delta} |f(v)|^p\,dm \right)^{1/p},$$

2.4. Corollary. *For $p \geq 1$ and $\lambda \in \partial\Omega$ we have*

$$(2.23) \qquad \dim L_a^p(\Omega)/[(z - \lambda)L_a^p(\Omega)]^- \leq 1.$$

Proof. Let $\lambda \in \partial\Omega$ and $p \geq 2$. Let $x_1, x_2 \in [(z - \lambda)L_a^p(\Omega)]^\perp$ and $x = x_2(1)x_1 - x_1(1)x_2$. Then for each $f \in A_\Omega^p(\lambda)$ we have $f = f(\lambda) + (z - \lambda)g$, $g \in L_a^p(\Omega)$, i.e. $x(f) = 0$. By Theorem 2.1 $x = 0$ and (2.23) follows. Analogously, if $1 \leq p < 2$ and $x_1, x_2 \in [(z - \lambda)L_a^p(\Omega)]^\perp$, we obtain that the functional $x = x_2[(z-\lambda)^{-1}]x_1 - x_1[(z-\lambda)^{-1}]x_2$ is in $[(z-\lambda)^{-1}A_\Omega^p(\lambda)]^\perp$, hence $x = 0$ which implies (2.23).

The result also holds for $\lambda \in \Omega$. The following proposition was proved in [3].

2.5. Proposition. *For $\lambda \in \Omega$ and $p \geq 1$, $(z - \lambda)L_a^p(\Omega)$ is a closed subspace of $L_a^p(\Omega)$ with codimension 1.*

3. INVARIANT SUBSPACES WITH FINITE CODIMENSION

Throughout the following Ω will be a bounded domain in $\mathbb{C}$. In order to describe the finite codimensional invariant subspaces of the operator M_z we use the following general method (see also [3] and [6]).

3.1. Lemma. *Let X be a normed linear space and $T: X \to X$ a bounded linear operator. Assume that for every $\lambda \in \mathbb{C}$ we have*

$$(3.1) \qquad \dim X/[(T - \lambda I)X]^- \leq 1.$$

Then each nontrivial closed invariant subspace E of T with finite codimension in X has the form

$$(3.2) \qquad E = [Q(T)X]^-,$$

where Q is a polynomial whose degree equals the codimension of E and whose zeros lie in the residual spectrum $\sigma_r(T)$.

Proof. Let E be a closed invariant subspace of T with $\dim X/E < \infty$ and define the linear operator T_1 on X/E by

$$(3.3) \qquad T_1(f + E) = Tf + E, \qquad f \in X.$$

Since E is invariant, T_1 is bounded and there exists a polynomial P of degree less than or equal to $\dim X/E$, such that $P(T_1) = 0$, or equivalently $[P(T)X]^- \subset E$. Let $P = QP_1$ with P, Q_1 polynomials having zeros in $\sigma_r(T)$ and $\mathbb{C}\backslash\sigma_r(T)$ respectively. Then obviously $[P(T)X]^- = [Q(T)X]^-$. Moreover, by (3.1) we have $\dim\ker(T^* - \lambda I) \leq 1$, $\lambda \in \mathbb{C}$, which implies

$$(3.5) \qquad \dim\ker Q(T)^* = \dim X/[Q(T)X]^- \leq \text{degree}\, Q,$$

by induction. Consequently

$$(3.6) \qquad \dim X/E \leq \dim X/[Q(T)X]^- \leq \text{degree}\, Q \leq \text{degree}\, P \leq \dim X/E,$$

which proves (3.2).

Lemma 3.1 may be applied to the operator M_z on $L_a^p(\Omega)$, $p \geq 1$. This follows by Corollary 2.4 and Proposition 2.5. We begin with the case $1 \leq p < 2$.

3.2. Theorem. *If E is a nontrivial closed invariant subspace of M_z with finite codimension in $L_a^p(\Omega)$, $1 \le p < 2$, then*

$$(3.7) \qquad\qquad\qquad E = QL_a^p(\Omega),$$

where Q is a polynomial whose degree equals the codimension of E and whose zeros lie in $\Omega \cup \partial_{p-r}(\Omega)$. Conversely, each subspace of the form (3.7) is a closed invariant subspace of M_z whose codimension equals the degree of Q.

Proof. In order to apply Lemma 3.1, we determine first $\sigma_r(M_z)$. It is well known that the spectrum of M_z equals Ω^-. If $\lambda \in \Omega$, $(z - \lambda)L_a^p(\Omega)$ is closed and has codimension 1, by Proposition 2.5. Let $\lambda \in \partial_{p-r}(\Omega)$. Then each $f \in L_a^p(\Omega)$ may be written as $f = (z - \lambda)^{-1}f_1$, $f_1 \in A_\Omega^p(\lambda)$. Let $f_1 = f_1(\lambda) + (z - \lambda)f_1'(\lambda) + (z - \lambda)^2 g$, $g \in A_\Omega^p(\lambda)$. Then

$$(3.8) \qquad\qquad f = (z - \lambda)^{-1}f_1(\lambda) + f_1'(\lambda) + (z - \lambda)g.$$

For $1 \le p < 2$ the constant functions are in $(z - \lambda)L_a^p(\Omega)$, hence $(z - \lambda)L_a^p(\Omega)$ has codimension 1 and is also closed, because it is the range of $M_z - \lambda I$. Finally, if $\lambda \in \partial_{p-e}(\Omega)$, let $\{\lambda_n\}$ be a sequence in $\partial\Omega \backslash \{\lambda\}$ tending to λ. Then

$$(z - \lambda_n)^{-1} = (z - \lambda)(z - \lambda)^{-1}(z - \lambda_n)^{-1} \in (z - \lambda)L_a^p(\Omega)$$

and $\{(z - \lambda_n)^{-1}\}$ converges weakly to $(z - \lambda)^{-1}$ in $L_a^p(\Omega)$. This follows using the Hahn-Banach theorem and the fact that for $q > 2$ the convolution of a compactly supported L^q-function and z^{-1} is a continuous function on $\mathbb{C}$. As above, we obtain that $(z - \lambda)^{-1}A_\Omega^p(\lambda)$ is contained in $[(z - \lambda)L_a^p(\Omega)]^-$, i.e. $\lambda \notin \sigma_r(M_z)$. We have shown that $\sigma_r(M_z) = \Omega \cup \partial_{p-r}(\Omega)$ and that for each $\lambda \in \sigma_r(M_z)$, $(M_z - \lambda I)L_a^p(\Omega)$ is closed and has codimension 1. Consequently, for every polynomial Q whose zeros lie in $\Omega \cup \partial_{p-r}(\Omega)$, $QL_a^p(\Omega)$ is a closed invariant subpsace whose codimension equals the degree of Q. Conversely, if E is such an invariant subspace, we obtain by Lemma 3.1, Corollary 2.4, and the above argument that E has the form (3.7).

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The structure of finite codimensional invariant subspaces of $L_a^p(\Omega)$ is more complicated in the case when $p \ge 2$. S. Axler and P. Bourdon proved in [3, Theorem 7] that there exist bounded domains Ω and closed invariant subspaces E of M_z with finite codimension in $L_a^2(\Omega)$ such that E cannot be written in the form $E = hL_a^2(\Omega)$ for any bounded analytic function h in Ω. On the other hand, it follows immediately from Corollary 2.4 and Lemma 3.1 that each such invariant subspace has the form $[QL_a^p(\Omega)]^-$, where Q is a polynomial. The reason for this complication is the fact that for $p \ge 2$ there exist domains Ω with essential boundary points λ for $L_a^p(\Omega)$ such that $(z - \lambda)L_a^p(\Omega)$ is not dense in $L_a^p(\Omega)$. Before stating our result on finite codimensional invariant subspaces we are going to analyze this situation more closely.

Note first that if $\lambda \in \Omega$ then $(z - \lambda)^k L_a^p(\Omega)$, $k \ge 1$, is closed and has codimension k, by Proposition 2.5. Moreover, it is easy to check that for $\lambda \in \Omega$, $[(z-\lambda)^k L_a^p(\Omega)]^\perp$ is spanned by the bounded linear functionals y_j, $0 \le j \le k - 1$, given by

$$(3.9) \qquad\qquad y_j(f) = f^{(j)}(\lambda), \qquad f \in L_a^p(\Omega),$$

where $f^{(0)} = f$ and $f^{(j)}$ is the jth derivative of f. The same is true for $\lambda \in \partial_{p-r}(\Omega)$, $p \geq 2$, as well. Indeed for such λ there exists a neighborhood V of λ such that every $f \in L_a^p(\Omega)$ extends analytically to $\Omega \cup V$. If V_1 is a connected neighborhood of λ with $V_1^- \subset V$, the inclusion map from $L_a^p(\Omega \cup V_1)$ to $L_a^p(\Omega)$ has a bounded inverse by the open mapping theorem and the assertion follows as above.

3.3. Proposition. *Let* $p \geq 2$ *and* $\lambda \in \partial_{p-e}(\Omega)$. *For each integer* $k \geq 1$ *we have*

$$(3.10) \qquad \dim[(z - \lambda)^k L_a^p(\Omega)]^\perp \leq k.$$

Moreover, if n *is the largest integer such that the linear functionals* x_j, $0 \leq j \leq n - 1$, *given by*

$$(3.11) \qquad x_j(f) = f^{(j)}(\lambda), \qquad f \in A_\Omega^p(\lambda),$$

extend to bounded linear functionals on $L_a^p(\Omega)$, *then* $[(z - \lambda)^k L_a^p(\Omega)]^\perp$ *has dimension* n *and is spanned by the functionals* x_j, $0 \leq j \leq n - 1$.

Proof. The inequality (3.10) is an immediate consequence of Corollary 2.4. We prove the second half of the statement by induction. If $k = 1$ and $x \in [(z - \lambda)L_a^p(\Omega)]^\perp$, $x \neq 0$, then for each $f \in A_\Omega^p(\lambda)$ write $f = f(\lambda) + (z - \lambda)g$, $g \in L_a^p(\Omega)$, which leads to

$$(3.12) \qquad x(f) = f(\lambda)x(1).$$

Thus $x(1) \neq 0$ and $x_0 = [x(1)]^{-1}x$ spans $[(z - \lambda)L_a^p(\Omega)]^\perp$ by Corollary 2.4. Now suppose that the assertion holds for some $k \geq 1$. If $(z - \lambda)^{k+1}L_a^p(\Omega)$ is dense in $(z - \lambda)^k L_a^p(\Omega)$ there is nothing to prove. Otherwise, for $0 \leq j \leq k$, $\dim[(z - \lambda)^j L_a^p(\Omega)]^-/[(z - \lambda)^{j+1}L_a^p(\Omega)] = 1$, hence $[(z - \lambda)^k L_a^p(\Omega)]^\perp$ has dimension k and is spanned by the bounded linear functionals x_j, $0 \leq j \leq k - 1$, given by (3.11). Further, if $x \in \ker(M_z^* - \lambda I)^{k+1} \setminus \ker(M_z^* - \lambda I)^k$, we have $(M_z^* - \lambda I)^k x \in \ker(M_z^* - \lambda I)$, that is

$$(3.13) \qquad (M_z^* - \lambda I)^k x = c \cdot x_0,$$

where $c \neq 0$ is a constant. Let

$$(3.14) \qquad x_k = c^{-1}k! \left[x - \sum_{j=0}^{k-1} \frac{1}{j!} x[(z - \lambda)^j] x_j \right].$$

Then $x_k \in [L_a^p(\Omega)]^*$ and for $f \in A_\Omega^p(\lambda)$,

$$(3.15) \qquad x_k(f) = c^{-1}k! x \left[f - \sum \frac{1}{j!} f^{(j)}(\lambda)(z - \lambda)^j \right] = f^{(k)}(\lambda),$$

which finishes the proof.

3.4. Remark. For $p \geq 2$ and $\lambda \in \partial\Omega$ let $N_p(\lambda) = \sup_{k \geq 1} \dim[(z - \lambda)^k L_a^p(\Omega)]^\perp$. Then $N_p(\lambda) \in \mathbb{N} \cup \{0, \infty\}$ and by Proposition 3.3 and its proof it follows that the codimension of $[(z - \lambda)^k L_a^p(\Omega)]^-$ equals $\min\{k, N_p(\lambda)\}$, for every integer $k \geq 0$. It is clear that $N_p(\lambda) = \infty$, if $\lambda \in \partial_{p-r}(\Omega)$. If $\lambda \in \partial_{p-e}(\Omega)$ it will be shown by examples in the next section that $N_p(\lambda)$ may be any nonnegative integer, or even ∞. The above definition makes sense for $1 \leq p < 2$, as well, but in this case $N_p(\lambda)$ is either 0 or ∞, according to $\lambda \in \partial_{p-e}(\Omega)$ or $\lambda \in \partial_{p-r}(\Omega)$, by Theorem 3.2.

We are now in position to state the characterization of finite codimensional invariant subspaces of $L_a^p(\Omega)$, $p \geq 2$.

3.5. Theorem. *Let $p \geq 2$ and let E be a nontrivial closed invariant subspace of M_z having finite codimension in $L_a^p(\Omega)$. Then there exists a polynomial Q whose zeros lie in Ω^- and whose degree equals the codimension of E such that*

$$(3.16) \qquad E = [QL_a^p(\Omega)]^-.$$

Furthermore, if

$$(3.17) \qquad Q = Q_1 \cdot \prod_{j=1}^n (z - \lambda_j)^{k_j},$$

where Q_1 is a polynomial whose zeros lie in $\Omega \cup \partial_{p-r}(\Omega)$, λ_j are distinct points in $\partial_{p-e}(\Omega)$, and k_j are positive integers, then $k_j \leq N_p(\lambda_j)$ and

$$(3.18) \qquad E = Q_1 \left[\prod_{j=1}^n (z - \lambda_j)^{k_j} L_a^p(\Omega) \right]^- = Q_1 \cdot \bigcap_{j=1}^n [(z - \lambda_j)^{k_j} L_a^p(\Omega)]^-.$$

Conversely, each subspace of the form (3.16) is a closed invariant subspace of M_z whose codimension equals

$$\text{degree } Q_1 + \sum_{j=1}^n \min\{k_j, N_p(\lambda_j)\},$$

where Q_1, λ_j, and k_j are given by (3.17).

For the proof we need the following simple lemma.

3.6. Lemma. *Let X be a normed linear space, let $T_1, T_2 \colon X \to X$ be bounded linear operators which commute and satisfy $\ker T_1^* \cap \ker T_2^* = \{0\}$. If $\dim \ker T_1^* < \infty$, then*

$$(3.19) \qquad (T_1 T_2 X)^- = (T_1 X)^- \cap (T_2 X)^-,$$

and

$$(3.20) \qquad \ker(T_1 T_2)^* = \ker T_1^* + \ker T_2^*.$$

Proof. Obviously $(T_1 T_2 X)^- \subset (T_1 X)^- \cap (T_2 X)^-$. Let $K = \ker T_1^*$ and note that $T_2^* K \subset K$ and that $T_2^*|K$ is injective. Since $\dim K < \infty$, it follows that $T_2^*|K$ is onto. Now let $x \in \ker(T_1 T_2)^* = \ker T_1^* T_2^*$. Then $T_2^* x \in K$, i.e.

$$(3.21) \qquad T_2^* x = T_2^* y, \qquad y \in K.$$

If $f \in (T_1 X)^- \cap (T_2 X)^-$, $f = \lim_{n \to \infty} T_2 f_n$, we have

$$x(f) = \lim_{n \to \infty} T_2^* x(f_n) = \lim_{n \to \infty} y(T_2 f_n) = 0,$$

which proves (3.19). From (3.21) we obtain also that $x - y \in \ker T_2^*$ and (3.20) follows.

Proof of Theorem 3.5. The existence of the polynomial Q follows by Lemma 3.1. Let $Q = Q_1 \cdot \prod_{j=1}^n (z - \lambda_j)^{k_j}$ with Q_1, λ_j, k_j as in (3.17). Since multiplication by Q_1 is a bounded linear operator with closed range on $L_a^p(\Omega)$ we obtain

$$(3.22) \qquad [QL_a^p(\Omega)]^- = Q_1 \left[\prod_{j=1}^n (z - \lambda_j)^{k_j} L_a^p(\Omega) \right].$$

Furthermore, if $j \neq s$, then the bounded linear operators given by multiplication by $(z - \lambda_j)^{k_j}$ and $(z - \lambda_s)^{k_s}$ satisfy the conditions in Lemma 3.6, hence

$$(3.23) \qquad \left[\prod_{j=1}^{n} (z - \lambda_j)^{k_j} L_a^p(\Omega) \right]^{-} = \bigcap_{j=1}^{n} [(z - \lambda_j)^{k_j} L_a^p(\Omega)]^{-}$$

$$= \bigcap_{j-1}^{n} [(z - \lambda_j)^{k_j'} L_a^p(\Omega)]^{-},$$

where $k_j' = \min\{k_j, N_p(\lambda_j)\}$, $1 \leq j \leq n$. Using (3.20), (3.22) and (3.21) we obtain

$$(3.24) \qquad \text{degree } Q = \text{degree } Q_1 + \sum_{j=1}^{n} k_j = \dim L_a^p(\Omega)/E$$

$$= \text{degree } Q_1 + \sum_{j=1}^{n} k_j'.$$

Thus $k_j = k_j'$ and the result follows. The converse is an immediate consequence of Lemma 3.6 and (3.23).

As pointed out in the introduction, the results of this section provide positive answers to the questions raised in [3] concerning finite codimensional invariant subspaces in $L_a^p(\Omega)$.

4. THE SPACES $(z - \lambda)^k L_a^p(\Omega)$

There is a nice connection between the spaces $(z - \lambda) L_a^p(\Omega)$, $\lambda \in \partial\Omega$ and the structure of the maximal ideal space of the Banach algebra $H^\infty(\Omega)$ of bounded analytic functions in Ω with the norm $\|f\|_\infty = \sup_{z \in \Omega} |f(z)|$. For $\lambda \in \partial\Omega$ denote by M_λ the set of complex homomorphisms x of $H^\infty(\Omega)$ satisfying $x(f) = f(\lambda)$ for each $f \in H^\infty(\Omega)$ which extends analytically in a neighborhood of λ.

4.1. Lemma. *If $p \geq 1$ and $\lambda \in \partial\Omega$ is such that $(z - \lambda) L_a^p(\Omega)$ is not dense in $L_a^p(\Omega)$, then M_λ contains a homomorphism which is w^*-continuous on $H^\infty(\Omega) \subset [L^1(\Omega, m)]^*$.*

Proof. A homomorphism x is w^*-continuous if and only if there exists $F \in L^1(\Omega, m)$ such that $x(h) = \int_\Omega hF \, dm$, $h \in H^\infty(\Omega)$. For $\lambda \in \partial_{p-r}(\Omega)$, M_λ contains only the point evaluation at λ which is w^*-continuous. If $p \geq 2$ and $\lambda \in \partial_{p-e}(\Omega)$, then by Proposition 3.3, there exists $x_0 \in [(z - \lambda) L_a^p(\Omega)]^\perp$ satisfying $x_0(f) = f(\lambda)$, $f \in A_\Omega^p(\lambda)$. Let $h \in H^\infty(\Omega)$ and define $x_h(f) = x_0(hf) - x_0(h)x_0(f)$, $f \in L_a^p(\Omega)$. Then $x_h \in [(z - \lambda) L_a^p(\Omega)]^\perp$ and $x_h(1) = 0$, i.e. $x_h = 0$ which leads to

$$(4.1) \qquad x_0(hf) = x_0(h)x_0(f).$$

In particular, x_0 is multiplicative on $H^\infty(\Omega)$, $x_0|H^\infty(\Omega)$ is w^*-continuous and belongs to M_λ.

According to results in [5], the existence of a w^*-continuous homomorphism in M_λ is equivalent to the fact that λ is not a peak point for $H^\infty(\Omega)$, i.e. there exists no $h \in H^\infty(\Omega)$ satisfying $\lim_{z \to \lambda} h(z) = 1$ and $\lim_{z \to \lambda'} \sup |h(z)| < 1$, for all $\lambda' \in \partial\Omega \setminus \{\lambda\}$. Then the results proved in §3 yield

4.2. Corollary. *Assume that each point of $\partial\Omega$ is a peak point for $H^\infty(\Omega)$. Let $p \geq 1$ and let E be a closed invariant subspace with finite codimension in $L_a^p(\Omega)$. Then there exists a polynomial Q whose degree equals the codimension of E and whose zeros lie in Ω such that $E = QL_a^p(\Omega)$.*

Proof. By Lemma 4.1 we have $\partial_{p-r}(\Omega) = \varnothing$ and for $p \geq 2$, $N_p(\lambda) = 0$ for all $\lambda \in \partial\Omega$. The result follows by Theorem 3.2 and Theorem 3.5.

Under the slightly more restrictive assumption that no connected component of $\partial\Omega$ reduces to a point, this result was proved by S. Axler and P. Bourdon in [3].

In what follows we are going to give some examples concerning the numbers $N_p(\lambda)$, $p \geq 2$, defined in the previous section. We shall consider domains of the type

$$(4.2) \qquad \Omega = \{z \colon 0 < |z| < 1\} \setminus \left(\bigcup_{n\geq 1} \Delta_n \right),$$

where Δ_n, $n \geq 1$, are disjoint closed discs contained in $\{z \colon 0 < |z| < 1\}$ whose centers $c_n > 0$ and radii $r_n > 0$ decrease to zero.

For such domains Ω we have $\partial_{p-e}(\Omega) = \partial\Omega$ because $\partial\Omega \setminus \{0\} \subset \partial_{p-e}(\Omega)$ and $\partial_{p-e}(\Omega)$ is a closed set. It is well known [5] that if $\sum_{n\geq 1} r_n/c_n < \infty$, 0 is not a peak point for $H^\infty(\Omega)$. As Corollary 4.2 shows this is a necessary condition in order to have $N_p(0) > 0$.

4.3. Example. Let k be a nonnegative integer and $p \geq 2$. If

$$(4.3) \qquad \lim_{n\to\infty} c_n^{k+1} r_n^{-1+2/p} = 0, \quad \text{for } p > 2,$$

or

$$(4.4) \qquad \lim_{n\to\infty} c_n^{k+1} \left(\log \frac{1}{r_n} \right)^{1/2} = 0, \quad \text{for } p = 2,$$

then $N_p(0) \leq k$.

Indeed, for $n \geq 1$ let $f_n = (z - c_n)^{-1}$. Then $f_n \in A_\Omega^p(0)$ and for $p > 2$

$$(4.5) \qquad \|f_n\|_p \leq \left(\int_{r_n < |u| < 2} |u|^{-p} dm \right)^{1/p} \leq (p-2)^{-1/p} r_n^{-1+2/p},$$

and similarily for $p = 2$

$$(4.6) \qquad \|f_n\|_2 \leq \left(\log \frac{1}{r_n} \right)^{1/2}.$$

On the other hand

$$(4.7) \qquad f_n^{(k)}(0) = k! c_n^{-k-1}, \qquad n \geq 1.$$

Then (4.3) and (4.4) show that the linear functional $f \to f^{(k)}(0)$, $f \in A_\Omega^p(0)$ cannot be bounded on $L_a^p(\Omega)$, hence $N_p(0) \leq k$ by Proposition 3.3.

The next example is based on the following technical lemma which is essentially due to S. Axler and P. Bourdon [3, Proof of Theorem 7].

4.4. Lemma. *Let $p \geq 2$ and Ω be a domain as above. Assume that the numbers r_n, c_n, satisfy*

$$(4.8) \qquad c_n - c_{n+1} - 2r_{n+1} > 4r_n, \qquad n \geq 1, \quad c_1 + 2r_1 < 1.$$

(i) *For $f \in L_a^p(\Omega)$, $k \geq 0$ and $-(1 + c_1 + 2r_1)/2 < u < 0$*

$$(4.9) \qquad f^{(k)}(u) = (-1)^k k! \cdot \frac{1}{2\pi i} \sum_{n \geq 0} \int_{\Gamma_n} \frac{f(v)}{(v - u)^{k+1}} dv,$$

where $\Gamma_0 = \{z: |z| < (1 + c_1 + 2r_1)/2\}$ and for $n \geq 1$, $\Gamma_n = \{z: |z - c_n| = 2r_n\}$.

(ii) *For $p > 2$ there exists a constant $a_p > 0$, depending only on p such that*

$$(4.10) \qquad \left| \int_{\Gamma_n} \frac{f(v)}{(v - u)^{k+1}} dv \right| \leq a_p \|f\|_p 2^{k+2} c_n^{-k-1} r_n^{1-2/p}, \qquad n \geq 1.$$

(iii) *If $p = 2$ and r_n, c_n also satisfy*

$$(4.11) \qquad c_n - c_{n+1} - 2r_{n+1} > 4r_n^\gamma, \qquad n \geq 1,$$

for some $\gamma \in (0, 1)$, then there exists $a_2 > 0$ such that
$$(4.12)$$
$$\left| \int_{\Gamma_n} \frac{f(v)}{(v - u)^{k+1}} dv \right| \leq a_2 \|f\|_2 (1 - \gamma)^{-1} \cdot 2^{k+1} c_n^{-k-1} \left(\log \frac{1}{r_n} \right)^{-1/2}, \qquad n \geq 1.$$

Proof. The formula (4.9) was proved in [3] for $p = 2$ and $k = 0$. For arbitrary k it follows by complex differentiation. The inequalities (4.10) and (4.12) follow also with the method used in [3]. Let $\rho_n = (c_n - c_{n+1} - 2r_{n+1})/2 > 2r_n$. For each $t \in (r_n, \rho_n)$ we have

$$(4.13) \qquad \left| \int_{\Gamma_n} \frac{f(v)}{(v - u)^{k+1}} dv \right| = \left| \int_0^{2\pi} f(c_n + te^{i\theta})(c_n + te^{i\theta} - u)^{-k-1} te^{i\theta} d\theta \right|$$

$$\leq 2^{k+1} c_n^{-k-1} \int_0^{2\pi} |f(c_n + te^{i\theta})| t \, d\sigma.$$

If $p > 2$, multiplying both sides of the above inequality by t^{-2}, integrating on (r_n, ρ_n) and using Hölder's inequality we obtain

$$(4.14) \qquad (2r_n)^{-1} \left| \int_{\Gamma_n} \frac{f(v)}{(v - u)^{k+1}} dv \right|$$

$$\leq 2^{k+1} c_n^{-k-1} \int_{\Gamma_n < |v - c_n| < \rho_n} |f(v)| \cdot |v|^{-2} dm$$

$$\leq a_p \|f\|_p r_n^{-2/p} c_n^{-k-1} \cdot 2^{k+1}.$$

The inequality (4.12) follows in a similar way. Multiplying by t^{-1} in (4.13) integrating or (r_n, ρ_n), using Hölder's inequality and the fact that $\rho_n > r_n^\gamma$ we obtain

$$(4.15) \qquad (1 - \gamma) \log \frac{1}{r_n} \left| \int_{\Gamma_n} \frac{f(v)}{(v - u)^{k+1}} dv \right|$$

$$\leq a_2 \|f\|_2 2^{k+1} c_n^{-k-1} \left(\log \frac{1}{r_n} \right)^{1/2}.$$

4.5. Example. Assume that the numbers r_n, c_n satisfy (4.8) if $p > 2$ and also (4.11) if $p = 2$. If

$$(4.16) \qquad \sum_{n \geq 1} r_n^{1-2/p} c_n^{-k-1} < \infty, \quad \text{for } p > 2,$$

or

$$(4.17) \qquad \sum_{n \geq 1} \left(\log \frac{1}{r_n} \right)^{-1/2} c_n^{-k-1} < \infty, \quad \text{for } p = 2,$$

then $N_p(0) \geq k + 1$. Indeed by (4.9) and (4.10) or (4.12) there exists $b_p > 0$, depending only on p, such that for $-(1 + c_1 + 2r_1)/2 < u < 0$ and $0 \leq j \leq k$

$$(4.18) \qquad |f^{(j)}(u)| \leq b_p j! 2^{j+1} \cdot \|f\|_p,$$

and the result follows by Proposition 3.3. If in addition r_n, c_n satisfy

$$(4.19) \qquad \lim_{n \to \infty} c_n^{-k-2} r_n^{1-2/p} = \infty, \quad \text{if } p > 2$$

or

$$(4.20) \qquad \lim_{n \to \infty} c_n^{-k-2} \left(\log \frac{1}{r_n} \right)^{-1/2} = \infty, \quad \text{if } p = 2,$$

then Example 4.3 shows that $N_p(0) = k + 1$. All these conditions are satisfied for $c_n = 2^{-1} n^{-2}$, $r_n = 5^{-1} n^{-\alpha}$, with $2k + 3 < \alpha < 2k + 4$, if $p > 2$, and for $c_n = 2^{-1} n^{-2}$, $r_n = 2^{-10} \exp(-n^\alpha)$, with $4k + 5 < \alpha < 4k + 8$, if $p = 2$. In this case we may take $\gamma = 1/2$. Finally if $c_n = 2^{-1} n^{-2}$, $r = 2^{-10} \exp(-a^n)$, $a > 1$, $n \geq 1$, it follows from above that $N_p(0) = \infty$ for all $p \geq 2$.

As Example 4.5 shows, the condition $N_p(\lambda) = \infty$ does not imply that λ is a removable boundary point for $L_a^p(\Omega)$, $p \geq 2$. This will follow under some additional conditions.

4.6. Proposition. Let $p \geq 2$ and $\lambda \in \partial \Omega$. The following are equivalent:
   (i) $\lambda \in \partial_{p-r}(\Omega)$.
   (ii) $(z - \lambda) L_a^p(\Omega)$ is closed.
   (iii) $N_p(\lambda) = \infty$ and there exists $c > 0$ such that the bounded linear functionals x_j given by (3.11) satisfy

$$(4.21) \qquad \|x_j\| \leq j! c^j \|x_0\|, \qquad j \geq 0.$$

Proof. We have seen in §3 that (i) implies (ii). If (ii) holds, then for every $f \in L_a^p(\Omega)$ we have

$$(4.22) \qquad \|f\|_p \leq a \|(z - \lambda) f\|_p,$$

for some positive constant a. Let x_j, $j \geq 0$, be the functionals given by (3.11). For $f \in A_\Omega^p(\lambda)$ let $f_0 = f$ and f_k be defined by

$$(4.23) \qquad f = \sum_{j=0}^{k-1} \frac{1}{j!} x_j(f)(z - \lambda)^j + (z - \lambda)^k f_k.$$

Then for $k \geq 1$

$$(4.24) \qquad f_{k-1} = x_0(f_{k-1}) + (z - \lambda) f_k,$$

hence

(4.25) $$\|f_k\|_p \le a(1 + \|x_0\|)\|f_{k-1}\|_p,$$

which leads to

(4.26) $$|x_k(f)| = k!|x_0(f_k)| \le k!\|x_0\| \, \|f_k\|,$$

and (4.21) follows with $c = a(1 + \|x_0\|)$. Finally, if (iii) holds, each $f \in A_\Omega^p(\lambda)$ is actually analytic in the disc centered at λ and of radius c^{-1}, and (i) follows by Theorem 2.1.

Thus if $p \ge 2$ and $\lambda \in \partial_{p-e}(\Omega)$ the subspaces $(z - \lambda)^k L_a^p(\Omega)$ are not closed. This holds for $1 \le p < 2$ too, in view of Theorem 3.2. The next result shows more than that, namely that $[(z - \lambda)^k L_a^p(\Omega)]^-$ cannot be written in the form $h \cdot L_a^p(\Omega)$ for any $h \in H^\infty(\Omega)$, unless h is invertible and $N_p(\lambda) = 0$. The proof is based on S. Axler's characterization of Fredholm multiplication operators on $L_a^p(\Omega)$ [1].

4.7. Proposition. *Let* $p \ge 1$, $\lambda \in \partial_{p-e}(\Omega)$, *and* k *be a positive integer. If* $h \in H^\infty(\Omega)$ *is such that*

(4.27) $$[(z - \lambda)^k L_a^p(\Omega)]^- = hL_a^p(\Omega),$$

then h *is invertible in* $H^\infty(\Omega)$ *and thus* $(z - \lambda)^k L_a^p(\Omega)$ *is dense in* $L_a^p(\Omega)$.

Proof. Let M_h be the bounded linear operator on $L_a^p(\Omega)$ given by $M_h(f) = hf$, $f \in L_a^p(\Omega)$. Note that h is invertible if and only if M_h is. If $1 \le p < 2$ and $\lambda \in \partial_{p-e}(\Omega)$, $[(z - \lambda)^k L_a^p(\Omega)]^- = L_a^p(\Omega)$, i.e. M_h is onto. M_h is also injective, hence invertible and the result follows. Let now $p \ge 2$. From (4.27) and Proposition 3.3 we obtain that M_h is a Fredholm operator on $L_a^p(\Omega)$. By Axler's theorem [1], $|h|$ is bounded away from zero near $\partial_{p-e}(\Omega)$. Then if $(z - \lambda)^k = hg$, $g \in L_a^p(\Omega)$, it follows that $g \in H^\infty(\Omega)$. Moreover, M_g must have dense range in $L_a^p(\Omega)$ by (4.27). Assume that $N_p(\lambda) > 0$ and let $x_0 \in [(z - \lambda)L_a^p(\Omega)]^\perp$ satisfy $x_0(f) = f(\lambda)$, $f \in A_\Omega^p(\lambda)$. By Lemma 4.1, $x_0|H^\infty(\Omega)$ belongs to M_λ, and

(4.28) $$0 = x_0[(z - \lambda)^k] = x_0(h)x_0(g),$$

which leads to $x_0(g) = 0$. An application of (4.1) shows that $x_0(gf) = x_0(g)x_0(f) = 0$, for all $f \in L_a^p(\Omega)$, i.e. $x_0 \in [gL_a^p(\Omega)]^\perp$ which contradicts the fact that M_g has dense range. Thus $N_p(\lambda) = 0$, and M_h is invertible, which finishes the proof.

REFERENCES

1. S. Axler, *Multiplication operators on Bergman spaces*, J. Reine Angew. Math. **336** (1982), 26–44.

2. S. Axler, J. B. Conway, and G. McDonald, *Toeplitz operators on Bergman spaces*, Canad. J. Math. **34** (1982), 466–483.

3. S. Axler and P. Bourdon, *Finite codimensional invariant subspaces of Bergman spaces*, Trans. Amer. Math. Soc. **306** (1988), 805–817.

4. T. W. Gamelin, *Uniform algebras*, Prentice-Hall, Englewood Cliffs, N.J., 1969.

5. T. W. Gamelin and J. Garnett, *Distinguished homomorphisms and fiber algebras*, Amer. J. Math. **92** (1970), 455–474.

6. R. Gellar, *Cyclic vectors and parts of the spectrum of a weighted shift*, Trans. Amer. Math. Soc. **146** (1969), 69–85.

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